國立臺北科技大學

九十三學年度資訊工程系碩士班入學考試

工程數學試題

填准考證號碼　　　　　　　　　　　　

注意事項：

1. 本試題共 5 題，配分共 100 分。
2. 請按順序標明題號作答，不必抄題。
3. 全部答案均須答在答案卷之答案欄內，否則不予計分。

Problem 1. (20%, 5% each) Write down the precise definition for each of the underlined terms in the context.

(a) A transformation $T: V \to W$ is <u>linear</u>.

(b) A scalar λ is an <u>eigenvalue</u> of an $n\times n$ matrix A.

(c) Two random variables X and Y are (statistically) <u>independent</u>.

(d) The <u>correlation coefficient</u> of two random variables X and Y.

Problem 2. (20%, 5% each) Determine the <u>invertibility</u> for each of the following matrices. Your answer should be 'invertiable', 'not invertible', or 'undetermined'. Give supportive arguments to your answer <u>with as few computations as possible</u>.

(a) The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 7 \\ 5 & 6 & 11 \end{bmatrix}$. (Avoid using determinants or row deduction.)

(b) A 2×2 transformation matrix B that stands for a projection onto the x-axis.

(c) An $n\times n$ lower triangular matrix C with $c_{nn} = 0$.

(d) An $n\times n$ matrix Q whose eigenvalues are all nonzero.

Problem 3. (20%, 5% each) Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, and a 4×4 matrix $U = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \mathbf{u}_4]$. Answer the following questions <u>with the least computations but sufficient evidence</u>.

(a) Show that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is an orthogonal set.

(b) Suppose that $\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$. Find c_2.

(c) Let $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Find the vector in W that is closest to $\mathbf{y}$.

(d) Find U^{-1}.

Problem 4. (20%, 5% each) Consider two discrete random variables X and Y with joint probability mass function $P\{X = 0, Y = -2\} = 1/2$ and $P\{X = 1, Y = 1\} = P\{X = -1, Y = 1\} = 1/4$.

(a) Find the marginal probabilities of X and Y.

(b) Are X and Y independent? Why?

(c) Calculate $E[X^2]$ and $E[XY]$.

(d) Are X and Y uncorrelated? Why?

Problem 5. (20%, 5% each) Let X be a random variable uniformly distributed in [-1,1]. Define another random variable $Y = X^2$.

(a) Plot the cumulative distribution function (c.d.f.) $F_X(x)$ of X.

(b) Evaluate the moment-generating function $M(t)$ of X.

(c) Calculate $E[Y]$.

(d) Calculate the probability density function (p.d.f.) of Y.